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Journal of Sound and Vibration 267 (2003) 1175–1183
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Letter to the Editor

A re-examination of geometric compatibility and shape function admissibility in Rayleigh–Ritz based substructure synthesis

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Received 22 October 2002; accepted 28 January 2003

1. Introduction

Substructure synthesis has often been plagued by the apparent constraints imposed on the choice of shape functions. Not only do most investigators choose to carry out a preliminary modal analysis of each substructure in order to pick the "best" shape functions (a.k.a. "component modes"), but very often the requirement of satisfying external geometric boundary conditions is imposed on these shape functions from the outset, and this restricts the choice or at the very least makes it problem-dependent. This paper intends to show that a much greater flexibility is in fact allowed: admissibility of underlying global shape functions (as required by the variational principle when a direction to the approximation on the eigenvalues is required) can be imposed at any stage of the process, and simply amounts to a series of constraints on the system variables, following from compliance with both the external and internal geometric boundary conditions (the latter often going under the name "geometric compatibility").

2. System equations

The study centres on free vibration of an undamped structure, described by the following self-adjoint differential eigenvalue form (following separation of time and space variables, and restricted in this case to a one-dimensional domain for the sake of simplicity, although extension to two- and three-dimensional systems is immediate) [1]:

$$\mathscr{L}w(x) = \lambda m(x)w(x), \quad 0 < x < L, \tag{1}$$

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0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00213-X

where $\mathscr{L}$ is a linear and homogeneous self-adjoint differential operator of even order $2p$, $w(x)$ is the displacement at position x of the structure (overall length L), λ is a parameter and $m(x)$ is the mass density at x.

The displacement is subject to $2p$ boundary conditions:

$$\mathscr{B}_i w(x) = 0, \quad i = 1, \ldots, p, \quad x = 0, L, \tag{2}$$

where $\mathscr{B}_i$ are linear and homogeneous differential operators of maximum order $2p-1$.

3. Variational approach

When writing the variational equivalent to the differential eigenvalue problem of order $2p$, the Rayleigh's quotient is defined as

$$R(\psi(x)) = \frac{\int_0^L \psi \mathscr{L} \psi \, \mathrm{d}x}{\int_0^L m\psi^2 \, \mathrm{d}x}$$

in which ψ is a trial function which must belong to the set of *comparison* functions, and must therefore

- satisfy all boundary conditions;
- be $2p$ times differentiable, i.e., be of class C^{2p-1} (continuous derivatives up to order $2p-1$).

In order to broaden the class of test functions, the numerator of this quotient is integrated by parts, while simultaneously expressing natural boundary conditions (BC): this results in replacing high-order derivatives by derivatives of maximum order $p-1$, and the final form of the numerator, defined as the energy inner product $[\psi, \psi]$, reads as

$$[\psi, \psi] = \int_0^L \sum_{k=0}^{p} a_k \left(\frac{\mathrm{d}^k \psi}{\mathrm{d}x^k} \right)^2 \mathrm{d}x + \sum_{m=0}^{p-1} b_m \left(\frac{\mathrm{d}^m \psi}{\mathrm{d}x^m} \right)^2 \Bigg|_0^L \tag{3}$$

for *self-adjoint systems*. This expression is quadratic and can be interpreted as twice the maximum potential energy of the system corresponding to the function ψ. The a_k and b_m coefficients are constants or space functions resulting from the expressed boundary conditions.

As can be deduced from Eq. (3), this expression of $[\psi, \psi]$ will provide the correct value of the potential energy for any function ψ which:

- satisfies the geometric boundary conditions, viz. those involving derivatives up to the order $p-1$;
- is p times differentiable, i.e., of class C^{p-1}.

It should be stressed at this point that the latter requirement holds over the *entire structure* under consideration. This means that ψ must exhibit continuous derivatives up to order $p-1$ over the whole structure.

When applying a Ritz procedure, ψ is expressed in terms of a limited size subspace spanned by independent shape functions $\psi_1, \ldots, \psi_n$:

$$\psi = \sum_{i=1}^{n} c_i \psi_i, \tag{4}$$

the c_i coefficients becoming the variables of the problem, and $[\psi, \psi]$ being expressed as

$$[\psi, \psi] = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} c_i c_j, \tag{5}$$

where from Eq. (3):

$$k_{ij} = [\psi_i, \psi_j] = \int_0^L \sum_{k=0}^{p} a_k \frac{\mathrm{d}^k \psi_i}{\mathrm{d}x^k} \frac{\mathrm{d}^k \psi_j}{\mathrm{d}x^k} \mathrm{d}x + \sum_{m=0}^{p-1} b_m \frac{\mathrm{d}^m \psi_i}{\mathrm{d}x^m} \frac{\mathrm{d}^m \psi_j}{\mathrm{d}x^m} \Bigg|_0^L, \tag{6}$$

leading to a "stiffness" matrix $\mathbf{K} = [k_{ij}]$.

Eq. (6) shows that the ψ_i functions must be admissible, i.e., satisfy the conditions stated above, and, in particular, be of class C^{p-1} over the *entire* structure, while also remaining independent.

4. Substructure synthesis

In substructure synthesis, shape functions are chosen for each separate substructure and the core of the technique, mainly as regards *precision* and hence *convergence*, lies in the choice of these shape functions (also inappropriately named component "modes", although they generally are *not* the solutions to any realistic eigenvalue problem!).

This has always been considered a critical phase of the procedure, and has led to much soul-searching and investigation (Refs. [1–7]). The choice is made either a priori ("assumed modes" or "Rayleigh–Ritz" approach), or following a detailed modal analysis (e.g., finite element) of the component resulting in a selection of the more pertinent modes for future use. In fact, as shown further on, requirements on these local shape functions will depend on how and when they are integrated into the procedure, and this is the key point the paper wishes to make.

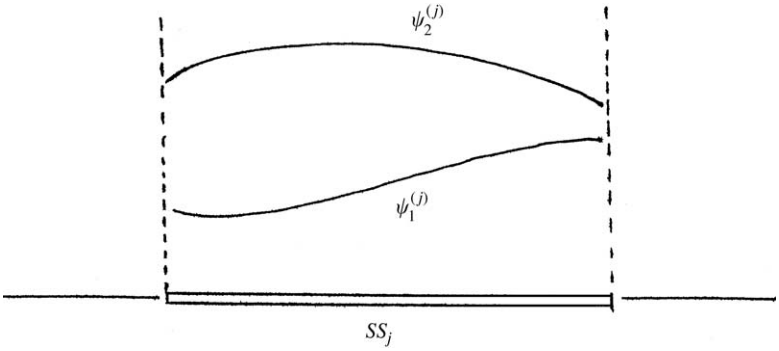

Fig. 1. Shape functions of a substructure.

It should be noted that, as these functions are defined only on a given substructure SS_j, they will generally exhibit some measure of discontinuity when considered over the entire structure. If one defines (see Fig. 1):

$$\psi_i^{(j)} \text{ as the } i\text{th shape function chosen on substructure } SS_j$$

$$\psi^{(j)} = \sum_i c_{ij}\psi_i^{(j)} \text{ as the local expansion on } SS_j$$

then

$$\psi_i^{(j)}, \psi^{(j)} \begin{cases} \neq 0 & \text{on } SS_j, \\ = 0 & \text{elsewhere,} \end{cases}$$

$$\psi \equiv \psi(j) \quad \text{on } SS_j$$

Even if compliance with external geometric boundary conditions is ensured (whether at the outset, or via later enforcement—see further on), these *local* shape functions will therefore generally *not* qualify as *global* admissible functions, which must be C^{p-1} over the entire structure. Local admissibility is not really useful as such; however, it goes part of the way towards global admissibility as will be seen.

This same issue is relevant in finite element modelling (FEM), where local and global shape functions are defined—this is not surprising as the FEM is a rather particular case of the Rayleigh–Ritz technique.

Global admissibility will be enforced via:

- compliance of $\psi = \sum_i \sum_j c_{ij}\psi_i^{(j)}$ with the external boundary conditions;
- an overall C^{p-1} condition on ψ.

Local admissibility will ensure C^{p-1} continuity of $\psi \equiv \psi^{(j)}$ within any SS_j, but enforcement of this same condition at the interfaces between substructures (“internal” boundaries) will result in a series of constraints on the c_{ij} coefficients, which amount to defining a new set of independent shape functions ϕ_i, which are

- defined over the entire structure,
- C^{p-1} over the entire structure,

as will be apparent in the simple examples detailed below.

This enforcement is generally (e.g., Refs. [4,5,7]) not considered in this light, but carried out under the heading of “geometric compatibility” (continuity of position ($p = 1$) or of slope and position ($p = 2$) depending on p) without any formal justification other than intuition.

5. Re-examination of global admissibility

The contention of this paper is that “geometric compatibility” is in fact a *necessary* requirement to ensure admissibility of the global shape functions. Physical continuity of shear force and bending moment at the interfaces could also be envisaged (and would lead to defining

independent *comparison* functions, i.e., of class C^{2p-1}, if all the external boundary conditions are also satisfied) but is more complicated to implement and is not required in order to satisfy the underlying variational principle; moreover, the results will not necessarily be better, since satisfaction of the internal and external boundary conditions is not the only issue: what is required is an overall satisfaction of the differential equations of motion, and some choices of overall shape functions are better than others in that respect, as demonstrated by Meirovitch and Kwak [6].

Compliance with *external* boundary conditions can also be enforced at the same stage (if the prior choice of local shape functions was such that this was not already ensured) and simply leads to additional constraints on the c_{ij}. This shows that there is no formal reason to treat internal and external boundary conditions differently or at different stages, as has often been the case (prior enforcement of external BC and later enforcement of internal BC via "compatibility"): both are necessary in the long run and can be enforced at any stage.

The global admissible shape functions ϕ_i never appear as such in the procedure, but are nonetheless mathematically prevalent, via the constraints. This again is exactly the same as in the FEM, where the overlapping in the assembled **M** and **K** matrices amounts to considering global admissible shape functions, and the compliance with external BC is achieved, at the same assembly stage, by dropping bordering lines and columns. Once again, substructuring and hierarchical FEM are simply different names for very similar techniques.

6. Some simple examples

The following simple examples will show how the global admissible functions result from the above considerations.

Consider first a Rayleigh–Ritz expansion for a single-structure problem (Fig. 2a) with external BC:

$$w(0) = 0 \tag{7}$$

and using three shape functions which do not meet this requirement separately, i.e.,

$$\psi = c_1\psi_1 + c_2\psi_2 + c_3\psi_3, \tag{8}$$

where $\psi_1(0) \neq 0,\ \psi_2(0) \neq 0,\ \psi_3(0) \neq 0$.

At some stage in the procedure, the external BC need to be enforced, and this results in the following linear constraint equation:

$$c_1\psi_1(0) + c_2\psi_2(0) + c_3\psi_3(0) \equiv 0, \tag{9}$$

which is an identity, allowing to express one of the variables c_i, say c_1, as

$$c_1 = -c_2\frac{\psi_2(0)}{\psi_1(0)} - c_3\frac{\psi_3(0)}{\psi_1(0)}, \tag{10}$$

whence

$$\begin{aligned}\psi &= c_2\left(\psi_2 - \frac{\psi_2(0)}{\psi_1(0)}\psi_1\right) + c_3\left(\psi_3 - \frac{\psi_3(0)}{\psi_1(0)}\psi_1\right)\\ &= c_2\phi_2 + c_3\phi_3\end{aligned} \tag{11}$$

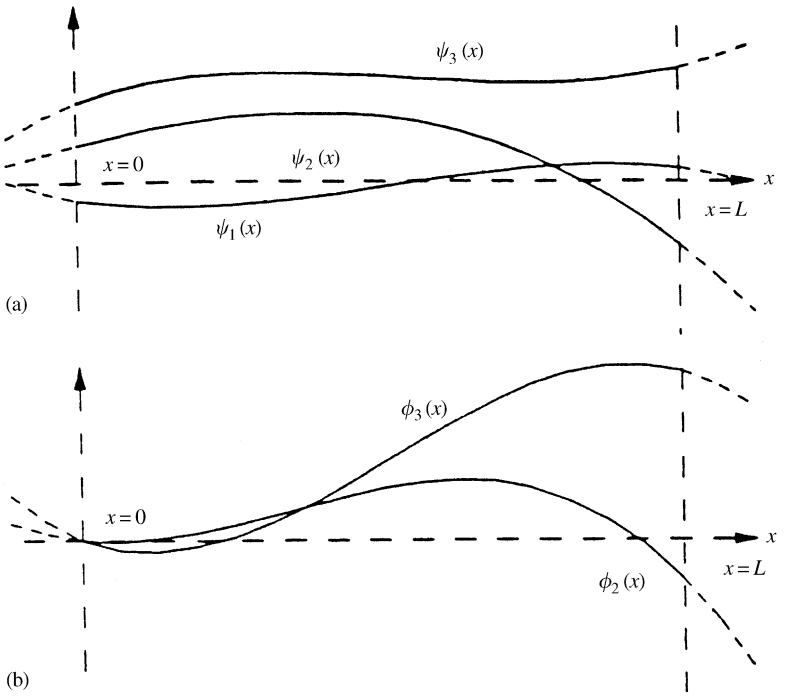

Fig. 2. (a) Single structure—non-admissible shape functions. (b) Single structure—resulting admissible shape functions.

and both ϕ_i functions, which are in fact the independent functions respectively related to c_2 and c_3, are such that

$$\phi_2(0) = \phi_3(0) = 0 \tag{12}$$

as can easily be ascertained and appears in Fig. 2b.

In a similar manner, compliance with internal BC between substructures will ensure that the remaining base functions (after co-ordinate reduction) are p times differentiable, which is also one of the admissibility requirements. To show this, consider two substructures SS_1 and SS_2 in a one-dimensional problem of differential order 2 (Fig. 3a). The geometric compatibility conditions at the interface point will require continuity of ψ, the latter being expanded via

$$\psi^{(1)}(x) = c_{11}\psi_1^{(1)}(x) + c_{21}\psi_2^{(1)}(x) \quad \text{on } SS_1\ (0<x<L),$$
$$\psi^{(2)}(y) = c_{12}\psi_1^{(2)}(y) + c_{22}\psi_2^{(2)}(y) \quad \text{on } SS_2\ (0<y<L'),$$

$$\left|\begin{array}{ccc} x=0 & SS_1 & x=L \\ & \psi_1^{(1)}(x) \text{ and } \psi_2^{(1)}(x) \neq 0 & \\ & = 0 \text{ elsewhere} & \end{array}\right| \left|\begin{array}{ccc} y=0 & SS_2 & y=L' \\ & \psi_1^{(2)}(y) \text{ and } \psi_2^{(2)}(y) \neq 0 & \\ & = 0 \text{ elsewhere} & \end{array}\right|$$

(i.e., all $\psi_i^{(j)}$ discontinuous when considered over the entire structure).

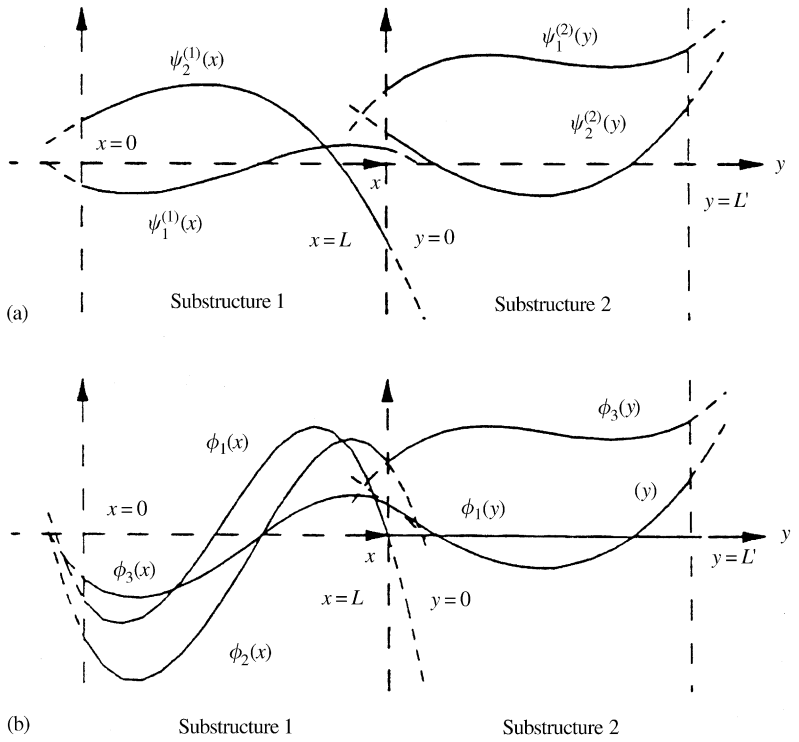

Fig. 3. (a) Two substructures—discontinuous shape functions. (b) Two substructures—resulting global admissible shape functions ($p = 1$).

Geometric compatibility at the interface ($x = L, y = 0$) amounts to $\psi^{(1)}(L) = \psi^{(2)}(0)$, i.e.,

$$c_{11}\psi_1^{(1)}(L) + c_{21}\psi_2^{(1)}(L) \equiv c_{12}\psi_1^{(2)}(0) + c_{22}\psi_2^{(2)}(0), \tag{13}$$

whence

$$c_{11} = \frac{c_{12}\psi_1^{(2)}(0) + c_{22}\psi_2^{(2)}(0) - c_{21}\psi_2^{(1)}(L)}{\psi_1^{(1)}(L)} \tag{14}$$

and

$$\begin{aligned}\psi &= c_{21}\left(\psi_2^{(1)}(x) - \frac{\psi_2^{(1)}(L)}{\psi_1^{(1)}(L)}\psi_1^{(1)}(x)\right) \\ &\quad + c_{12}\left(\psi_1^{(2)}(y) + \frac{\psi_1^{(2)}(0)}{\psi_1^{(1)}(L)}\psi_1^{(1)}(x)\right) \\ &\quad + c_{22}\left(\psi_2^{(2)}(y) + \frac{\psi_2^{(2)}(0)}{\psi_1^{(1)}(L)}\psi_1^{(1)}(x)\right) \\ &= c_{21}\phi_1 + c_{12}\phi_2 + c_{22}\phi_3. \end{aligned} \tag{15}$$

The resulting base functions ϕ_i are p times differentiable (in this case $p = 1$), as can readily be checked and appears in Fig. 3b.

7. Flexible choice of base functions

The above observations leave the user free to choose an initial set of base functions which are the most amenable to analytic manipulation and, moreover, an invariable set in such a way as to render the setting up of the **K** and **M** matrices totally *systematic*, i.e., problem-independent.

In the author's own experience (Refs. [8–12]), power series have most of the required advantages, as they are

- a user-independent and problem-independent choice;
- easy to implement and expand to a larger set;
- capable of satisfying any boundary condition, whether external or internal.

They fit the definition of quasi-comparison functions as coined by Meirovitch and Kwak [6], since they result from an a priori choice and do *not* constitute a "family" or "set" as defined in Refs. [6,7], viz., resulting from the exact solution to a simplified problem (generally as regards physical BC) and less flexible as regards their actual fit to the true behaviour of the system.

8. Conclusion

The author hopes to have demonstrated that the oft-used notion of geometric compatibility in modal synthesis is in fact equivalent to generating underlying admissible shape functions, which consequently exhibit this admissibility throughout the entire structure.

These global admissible functions never appear as such in the process, yet are present via the geometric constraints imposed on the system variables.

The fact that admissibility constraints, whether relating to external or internal boundaries, can thus be imposed at any stage of the process, makes the initial choice of base functions much more flexible and problem-independent, and therefore more amenable to automatic and symbolic computation. This has been amply demonstrated in the various references.

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